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The Lebesgue Function and Lebesgue Constant of Lagrange Interpolation for Erdős Weights

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We establish pointwise as well as uniform estimates for Lebesgue functions associated with a large class of Erdős weights on the real line. An Erdős weight is of the form $W := \exp(-Q)$, where $Q: \mathbf{R} \to \mathbf{R}$ is even and is of faster than polynomial growth at infinity. The archetypal examples are

$$W_{k,\alpha}(x) := \exp(-Q_{k,\alpha}(x)), \tag{i}$$

where $Q_{k,\alpha}(x) := \exp_k(|x|^\alpha)$, $\alpha > 1$, $k \geqslant 1$. Here $\exp_k := \exp(\exp(\exp(...)))$ denotes the kth iterated exponential.

$$W_{A,B}(x) := \exp(-Q_{A,B}(x)), \tag{ii}$$

where $Q_{A,B}(x) := \exp(\log(A + x^2))^B$, $B > 1$ and $A > A_0$. For a carefully chosen system of nodes $\chi_n := \{\xi_1, \xi_2, ..., \xi_n\}$, $n \geqslant 1$, our result imply in particular, that the Lebesgue constant $\|\Lambda_n(W_{k,\alpha}, \chi_n)\|_{L_\infty(\mathbf{R})} := \sup_{x \in \mathbf{R}} |\Lambda_n(W_{k,\alpha}, \chi_n)|\,(x)$ satisfies uniformly for $n \geqslant N_0$, $\|\Lambda_n(W_{k,\alpha}, \chi_n)\|_{L_\infty(\mathbf{R})} \sim \log n$. Moreover, we show that this choice of nodes is optimal with respect to the zeros of the orthonormal polynomials generated by W^2. Indeed, let $U_n := \{x_{j,n} : 1 \leqslant j \leqslant n\}$, $n \geqslant 1$, where the $x_{k,n}$ are the zeros of the orthogonal polynomials $p_n(W^2, \cdot)$ generated by W^2. Then in particular, we have uniformly for $n \geqslant N$, $\|\Lambda_n(W_{k,\alpha}, U_n)\|_{L_\infty(\mathbf{R})} \sim n^{1/6}(\prod_{j=1}^k \log_j n)^{1/6}$. Here, $\log_j := \log(\log(\log(...)))$ denotes the jth iterated logarithm. We deduce sharp theorems of uniform convergence of weighted Lagrange interpolation together with rates of convergence. In particular, these results apply to $W_{k,\alpha}$ and $W_{A,B}$. © 1998 Academic Press

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we investigate Lebesgue bounds and uniform convergence of Lagrange interpolation for Erdős weights. We recall that an Erdős weight has the form

$$W := \exp(-Q),$$

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where $Q: \mathbf{R} \to \mathbf{R}$ is even and is of faster than polynomial growth at infinity. The archetypal examples are

$$\text{(i)} \quad W_{k,\alpha}(x) := \exp(-Q_{k,\alpha}(x)), \tag{1.1}$$

where

$$Q_{k,\alpha}(x) := \exp_k(|x|^{\alpha}), \qquad k \geqslant 1, \alpha > 1.$$

Here $\exp_k := \exp(\exp(\exp(...)))$ denotes the kth iterated exponential.

$$\text{(ii)} \quad W_{A,\beta}(x) := \exp(-Q_{A,B}(x)), \tag{1.2}$$

where

$$Q_{A,B}(x) := \exp(\log(A + x^2))^B,$$
$$B \geqslant 1 \text{ and } A \text{ is large enough but fixed.}$$

Throughout, let $f: \mathbf{R} \to \mathbf{R}$ be continuous and satisfy the decay condition,

$$\lim_{|x| \to \infty} |fW|(x) = 0. \tag{1.3}$$

We set

$$E_n[f]_{W,\infty} := \inf_{P \in \mathscr{P}_n} \|(f-P)(x)\, W(x)\|_{L_\infty(\mathbf{R})} \tag{1.4}$$

to be the error of best weighted polynomial approximation to f from $\mathscr{P}_n$, $n \geqslant 1$.

Here, $\mathscr{P}_n$ denotes the class of polynomials of degree $\leqslant n$.

It is well known [9] that

$$E_n[f]_{W,\infty} \to 0 \qquad \text{as} \quad n \to \infty.$$

Now let

$$\chi_n := \{\xi_1, \xi_2, ..., \xi_n\}, \qquad n \geqslant 1,$$

be an arbitrary set of nodes. The Lagrange interpolation polynomial to f with respect to χ_n is denoted by $L_n[f, W, \chi_n]$. Thus, if

$$l_{j,n}(\chi_n) \in \mathscr{P}_{n-1}, \qquad 1 \leqslant j \leqslant n,$$

are the fundamental polynomials of Lagrange interpolation at ξ_j, $1 \leqslant j \leqslant n$, satisfying

$$l_{j,n}(\chi_n)(\xi_{j,n}) = \delta_{j,k}, \qquad 1 \leqslant k \leqslant n,$$

then

$$L_n[f, W, \chi_n](x) = \sum_{j=1}^{n} f(\xi_{j,n})\, l_{j,n}(\chi_n)(x) \in \mathcal{P}_{n-1}. \tag{1.5}$$

Now write

$$\begin{aligned} &\| W(f - L_n[f, W, \chi_n]) \|_{L_\infty(\mathbf{R})} \\ &\quad \leqslant E_{n-1}[f]_{W,\infty} \left(1 + \left\| W(x) \sum_{j=1}^{n} |l_{j,n}(\chi_n)(x)|\, W^{-1}(\xi_j) \right\|_{L_\infty(\mathbf{R})} \right) \\ &\quad = E_{n-1}[f]_{W,\infty} (1 + \| \Lambda_n(W, \chi_n) \|_{L_\infty(\mathbf{R})}), \end{aligned} \tag{1.6}$$

where $\| \Lambda_n(W, \chi_n) \|_{L_\infty(\mathbf{R})}$ is called the Lebesgue constant with respect to the weight W and the set of nodes χ_n, and $\Lambda_n(W, \chi_n)$ is the corresponding Lebesgue function.

Using (1.6), we see that estimates of the size of the Lebesgue constant enable one to deduce theorems on uniform convergence of Lagrange interpolation. As the subject of weighted Lagrange interpolation is an extensively researched and widely studied subject, we refer the reader to [1, 5–7, 10–15].

Now given a weight $W: \mathbf{R} \to (0, 1]$ as above, we may define orthonormal polynomials

$$p_n(x) := p_n(W^2, x) = \gamma_n x^n + \cdots, \qquad \text{with} \quad \gamma_n = \gamma_n(W^2) > 0,$$

satisfying

$$\int_{\mathbf{R}} p_n(W^2, x)\, p_m(W^2, x)\, W^2(x)\, dx = \delta_{mn}.$$

We denote the zeros of p_n by

$$-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty.$$

Put

$$U_n := \{ x_{j,n} : 1 \leqslant j \leqslant n \}, \qquad n \geqslant 1. \tag{1.7}$$

To formulate our results, we need a suitable class of Erdős weights from [8].

DEFINITION 1.1. Let $W := \exp(-Q)$, where $Q: \mathbf{R} \to \mathbf{R}$ is even, continuous, Q'' exists in $(0, \infty)$, $Q^{(j)} \geqslant 0$ in $(0, \infty)$, $j = 0, 2$, $Q^{(1)} > 0$ in $(0, \infty)$, and the function

$$T(x) := 1 + \frac{xQ''(x)}{Q(x)} \tag{1.8}$$

is increasing in $(0, \infty)$ with

$$\lim_{x \to \infty} T(x) = \infty, \qquad T(0^+) := \lim_{x \to 0^+} T(x) > 1. \tag{1.9}$$

Moreover, we assume that for some $C_j > 0$, $1 \leqslant j \leqslant 3$,

$$C_1 \leqslant \frac{T(x)}{(xQ'(x)/Q(x))} \leqslant C_2, \qquad x \geqslant C_3 \tag{1.10}$$

and for every $\varepsilon > 0$,

$$T(x) = O((Q(x))^{\varepsilon}), \qquad x \to \infty. \tag{1.11}$$

Then, we write $W \in \mathscr{E}$.

The principle examples of $W \in \mathscr{E}$ are $W_{k,\alpha}$ and $W_{A,B}$ given by (1.1) and (1.2), respectively. For more on this subject we refer the reader to [2–4, 8].

To state our results, we need some more notation.

We need the Mhaskar–Rakhmanov–Saff number a_u defined as the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)\, dt}{\sqrt{1 - t^2}}, \qquad u > 0. \tag{1.12}$$

Here, a_u exists and is a strictly increasing function of u [8, 9]. Among its uses is the infinite-finite range inequality

$$\|PW\|_{L_\infty(\mathbf{R})} = \|PW\|_{L_\infty[-a_n, a_n]}, \qquad P \in \mathscr{P}_n. \tag{1.13}$$

Note that a_n depends only on the degree of the polynomial P and not on P itself.

Now choose $y_0 \in [-a_n, a_n]$ so that

$$|p_n W(y_0)| = \|p_n W\|_{L_\infty(\mathbf{R})}. \tag{1.14}$$

As W is even, we may assume that $y_0 \geqslant 0$. We will show later that in fact $y_0 > 0$ and is very "close" to a_n. Fix y_0 as above.

Finally set

$$\delta_n := (nT(a_n))^{-2/3}, \qquad n \geqslant 1, \tag{1.15}$$

and

$$\Psi_n(x) := \begin{cases} \max\left\{\sqrt{1-\frac{|x|}{a_n}+L\delta_n}\,,\ \frac{1}{T(a_n)\sqrt{1-(|x|/a_n)+L\delta_n}}\right\}, \\ \qquad |x| \leqslant a_n \\ \Psi(a_n), \qquad |x| \geqslant a_n\,. \end{cases} \tag{1.16}$$

Here, $L > 0$ is fixed, but large enough throughout.

For more on these special sequences of functions, we refer the reader to [5, 8].

Here and throughout, for real sequences A_n and $B_n \not\equiv 0$,

$$A_n = O(B_n), \qquad A_n \sim B_n, \qquad \text{and} \qquad A_n = o(B_n)$$

will mean respectively that there exist constants $C_j > 0$, $j = 1, 2, 3$, independent of n, such that

$$\frac{A_n}{B_n} \leqslant C_1, \qquad C_2 \leqslant \frac{A_n}{B_n} \leqslant C_3, \qquad \text{and} \qquad \lim_{n\to\infty}\left|\frac{A_n}{B_n}\right| = 0.$$

Similar notation will be used for functions and sequences of functions.

Bounds for Lebesgue Constants and Uniform Convergence of Lagrange Interpolation for U_n, $n \geqslant 1$. We begin our investigation with the sequence of nodes, U_n, $n \geqslant 1$, defined by (1.7).

We prove:

THEOREM 1.2. *Let $W \in \mathscr{E}$. Then, uniformly for $n \geqslant N_0$,*

$$\|\Lambda_n(W, U_n)\|_{L_\infty(\mathbf{R})} \sim n^{1/6} T(a_n)^{1/6}. \tag{1.17}$$

In particular, given $\varepsilon > 0$, there exists $C > 0$ independent of n such that

$$\|\Lambda_n(W, U_n)\|_{L_\infty(\mathbf{R})} \leqslant Cn^{1/6+\varepsilon}.$$

We deduce:

COROLLARY 1.3. *Let $W \in \mathscr{E}$ and $r \geqslant 1$. Then there exists $C_j > 0$; $j = 1, 2$, independent of n and f so that for $n \geqslant N_0$,*

$$\begin{aligned} \text{(a)} \quad & \|(f - L_n[f, W, U_n]) W\|_{L_\infty(\mathbf{R})} \\ & \leqslant C_1 E_{n-1}[f]_{W,\infty}\, n^{1/6} T(a_n)^{1/6} \\ & \leqslant C_2 \omega_{r,\infty}\left(f, W, \frac{a_n}{n}\right) n^{1/6} T(a_n)^{1/6}. \end{aligned} \tag{1.18}$$

Here,

$$\omega_{r,\infty}(f, W, t) := [\sup_{0<h\leqslant t} \|W \Delta^r_{h\Phi_t^{1/2}(x)}(f)\|_{L_\infty(|x|\leqslant\sigma(2t))} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P) W\|_{L_\infty(|x|\geqslant\sigma(4t))}], \qquad t>0$$

is the weighted modulus of smoothness of f,

$$\sigma(t) := \inf\left\{a_u : \frac{a_u}{u} \leqslant t\right\}, \tag{1.19}$$

$$\Phi_t(x) := \left|1 - \frac{|x|}{\sigma(t)}\right| + T(\sigma(t))^{-1}, \qquad x \in \mathbf{R}, \tag{1.20}$$

and for an interval J and $h>0$,

$$\Delta^r_h(f, x, J) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + \frac{rh}{2} - ih\right), & x \pm \frac{rh}{2} \in J \\ 0, & \text{otherwise} \end{cases}.$$

(b) *Moreover, if f satisfies $f^{(r)} W \in L_\infty(\mathbf{R})$, then given $\varepsilon>0$,*

$$\|(f - L_n[f, W, U_n]) W\|_{L_\infty(\mathbf{R})} \leqslant C_3 \left(\frac{a_n}{n}\right)^r n^{1/6} T(a_n)^{1/6} \tag{1.21}$$

$$\leqslant C_3 n^{1/6+\varepsilon-r}. \tag{1.22}$$

Here $C_3>0$ is independent of n.

Thus we can ensure uniform convergence for every $r \geqslant 1$.

Remark. It is instructive at this point to recall that for $Q = Q_{k,\alpha}$ of (1.1),

$$T(a_n) = \prod_{j=1}^{k} \log_j n.$$

Moreover, in general, given $\varepsilon > 0$ and $n \geqslant 1$,

$$T(a_n) = O(n^{\varepsilon}).$$

(See also (2.7)). We thus observe that we may dispense with the $T(a_n)^{1/6}$ on the right hand side of (1.17) by inserting an extra weighting factor into the left hand side of (1.17) in the following sense.

Under the hypotheses of Theorem 1.2, we have uniformly for $n \geqslant N_0$,

$$\left\| \Lambda_n(W, U_n) \left(\left| 1 - \frac{|x|}{a_n} \right| + T(a_n)^{-1} \right)^{1/6} \right\|_{L_\infty(\mathbf{R})} \sim n^{1/6}. \tag{1.23}$$

This follows easily using the proof of (1.17) and (2.11).

A Better Behaving Lebesgue Function. We observe that although (1.21) yields uniform convergence for every $r \geqslant 1$, we can substantially improve our results, by choosing our interpolation points more carefully. The idea, first exploited by J. Szabados [14] for Freud weights on the real line, is motivated by (1.13). Recalling the definition of y_0 in (1.14) and U_n in (1.7), we set

$$V_{n+2} := \{-y_0, y_0\} \cup U_n, \qquad n \geqslant 1,$$

and prove:

THEOREM 1.4. *Let $W \in \mathscr{E}$. Then uniformly for $n \geqslant N_0$,*

$$\|\Lambda_{n+2}(W, V_{n+2})\|_{L_\infty(\mathbf{R})} \sim \log n. \tag{1.24}$$

Thus, by adding two completely new points of interpolation, we can achieve the much better order $\log n$ in comparison to the order $(nT(a_n))^{1/6}$ that we obtained merely using the zeros of p_n.

We deduce,

COROLLARY 1.5. *Let $W \in \mathscr{E}$ and $r \geqslant 1$. Then there exists $C_j > 0$, $j = 1, 2$ independent of f and n so that for $n \geqslant N_0$,*

$$\begin{aligned} \text{(a)} \quad & \|(f - L_{n+1}[f, W, V_{n+2}])\, W\|_{L_\infty(\mathbf{R})} \\ & \leqslant C_1 E_n[f]_{W, \infty} \log n \\ & \leqslant C_2 \omega_{r, \infty}\left(f, W, \frac{a_n}{n}\right) \log n. \end{aligned} \tag{1.25}$$

(b) *Moreover, if* f *satisfies* $f^{(r)}W \in L_\infty(\mathbf{R})$ *then, given* $\varepsilon > 0$,

$$\|(f - L_n[f, W, U_n])\, W\|_{L_\infty(\mathbf{R})} \leqslant C_3 \left(\frac{a_n}{n}\right)^r \log n \tag{1.26}$$

$$\leqslant C_3 n^{-r+\varepsilon} \log n. \tag{1.27}$$

Here $C_3 > 0$ *is independent of* n.

Remark. A natural question arises as to whether (1.24) holds (in a lower bound sense) for any system of nodes, at least for some Erdős weight. This and related questions will be considered in a future paper.

Pointwise Estimates for $\Lambda_n(W, U_n)$. We present pointwise estimates for $\Lambda_n(W, U_n)$. We emphasize our results and briefly sketch their proofs in Section 5 as the arguments are straightforward, but rather lengthy.

THEOREM 1.6. *Let* $W \in \mathscr{E}$.

(a) *Then for* $n \geqslant N_0$, *there exists* $C > 0$ *such that for* $|x| \leqslant a_n(1 + (L/2)\,\delta_n)$,

$$\Lambda_n(W, U_n)(x) \leqslant C[1 + \sqrt{a_n}\, |p_n W|(x) \times \left[\left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{1/4} \log\left(\frac{n(1 - (|x|/a_n) + L\delta_n)}{\Psi_n(x)}\right) + 1\right]\Bigg]. \tag{1.28}$$

Moreover, we have uniformly for $|x| \leqslant x_{1,n}$ *and* n,

$$\Lambda_n(W, U_n)(x) \sim 1 + \sqrt{a_n}\, |p_n W|\,(x) \times \left[\left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{1/4} \log\left(\frac{n(1 - (|x|/a_n) + L\delta_n)}{\Psi_n(x)}\right) + 1\right]\Bigg]. \tag{1.29}$$

(b) *Uniformly for* $n \geqslant N_0$ *and* $a_n(1 + (L/2)\,\delta_n) \leqslant |x| \leqslant 2a_n$,

$$\Lambda_n(W, U_n)(x) \sim \sqrt{a_n}\, |p_n W|\,(x)[1 + \delta_n^{1/4}]. \tag{1.30}$$

(c) *Uniformly for* $n \geqslant N_0$ *and* $|x| \geqslant 2a_n$,

$$\Lambda_n(W, U_n)(x) \sim \frac{a_n^{3/2}\, |p_n W|\,(x)}{|x|}\,[1 + \delta_n^{1/4}]. \tag{1.31}$$

Structure of This Paper. We close this section with some notation and remarks concerning the structure of this paper. Throughout, C, C_1, $C_2 \cdots > 0$ will denote constants independent of n, x and $P \in \mathscr{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L.

This paper is organized as follows.

In Section 2, we present our technical lemmas. In Section 3, we present the proofs of our upper bounds for (1.17) and (1.24). In Section 4, we prove Theorems 1.2 and 1.4 and Corollaries 1.3 and 1.5. Finally in Section 5, we sketch briefly the main ideas in the proof of Theorem 1.6.

2. TECHNICAL LEMMAS

LEMMA 2.1. *Let $W \in \mathscr{E}$ and set*

$$x_{0,n} := x_{1,n}(1 + L\delta_n) \qquad \text{and} \qquad x_{n,n+1} := -x_{0,n}.$$

(a) *There exists $A > 0$ independent of n and L such that for $n \geqslant 1$,*

$$\left| \frac{x_{1,n}}{a_n} - 1 \right| \leqslant A\delta_n. \tag{2.1}$$

(b) *Uniformly for $n \geqslant 2$ and $0 \leqslant j \leqslant n-1$,*

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n} \Psi_n(x_{j,n}). \tag{2.2}$$

(c) *Uniformly for $n \geqslant 2$ and $0 < j \leqslant n-1$,*

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n \tag{2.3}$$

and

$$\Psi_n(x_{j,n}) \sim \Psi_n(x_{j+1,n}). \tag{2.4}$$

(d) *For $n \geqslant 1$,*

$$\sup_{x \in \mathbf{R}} |p_n W|(x) \left| 1 - \frac{|x|}{a_n} \right|^{1/4} \sim a_n^{-1/2} \tag{2.5}$$

and

$$\sup_{x \in \mathbf{R}} |p_n W|(x) \sim n^{1/6} T(a_n)^{1/6} a_n^{-1/2}. \tag{2.6}$$

Proof. This is part of Lemma 2.1 of [5]. ∎

Now fix A in (2.1).

LEMMA 2.2. *Let* $W \in \mathscr{E}$.

(a) *Given* $\varepsilon > 0$ *and* $n \geqslant 1$, *there exists* $C > 0$ *independent of* n *such that*

$$a_n \leqslant Cn^{\varepsilon}, \qquad T(a_n) \leqslant Cn^{\varepsilon}, \qquad \text{and} \qquad \delta_n \leqslant CT(a_n)^{-\varepsilon}. \tag{2.7}$$

(b) *Given* $0 < \alpha < \beta$, *we have uniformly for* $n \geqslant C$,

$$T(a_{\alpha n}) \sim T(a_{\beta n}). \tag{2.8}$$

(c) *Uniformly for* $u \in (C, \infty)$ and $v \in [u/2, 2u]$, *we have*

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| T(a_n)^{-1}. \tag{2.9}$$

(d) *Given* $m \in \mathbf{N}$ *and* $n \geqslant N_0$, *we have for every* $\{P_k\}_{k=1}^{m} \in \mathscr{P}_n$

$$\left\| W \sum_{k=1}^{m} |P_k| \right\|_{L_\infty(\mathbf{R})} = \left\| W \sum_{k=1}^{m} |P_k| \right\|_{L_\infty[-a_n, a_n]}. \tag{2.10}$$

Moreover, given $r > 1$, there exists $C = C(r) > 0$ independent of n, m, and P_k such that

$$\begin{aligned} &\left\| W \left(\left| 1 - \frac{|x|}{a_n} \right| + T(a_n)^{-1} \right)^{1/6} \sum_{k=1}^{m} |P_k| \right\|_{L_\infty(\mathbf{R})} \\ &\quad \leqslant C \left\| W \left(\left| 1 - \frac{|x|}{a_n} \right| + T(a_n)^{-1} \right)^{1/6} \sum_{k=1}^{m} |P_k| \right\|_{L_\infty[-a_{r(n+1)}, a_{r(n+1)}]}. \end{aligned} \tag{2.11}$$

Proof. Parts (a)–(c) are found in Lemma 2.3 of [5], (2.10) follows as in Lemma 1 of [14], and then (2.11) follows using (2.10) and the method of Lemma 3.3 in [3]. ∎

Our next lemma establishes how "close" y_0 is to a_n.

LEMMA 2.3. *Let* $W \in \mathscr{E}$, $n \geqslant N_0$, *and* y_0 *be as in* (1.14). *Then, we have*

$$a_n(1 - B\delta_n) \leqslant y_0 \leqslant a_n \tag{2.12}$$

for some $B > 0$ *independent of* n *and* L.

Proof. By (2.5), (2.6), and the definition of δ_n (see (1.15)), there exist $C_j > 0$, $j = 1, 2$ such that

$$C_1 a_n^{-1/2} (nT(a_n))^{1/6} \leqslant |p_n(y_0)| \, W(y_0) \leqslant C_2 a_n^{-1/2} \min\left\{ \left|1 - \frac{y_0}{a_n}\right|^{-1/4}, \delta_n^{-1/4} \right\}. \tag{2.13}$$

Then, this gives

$$\max\left\{ \left|1 - \frac{y_0}{a_n}\right|, \delta_n \right\} \leqslant C_3 \, \delta_n. \tag{2.14}$$

Now by the definition of y_0, we have clearly that $y_0 \leqslant a_n$. Moreover, if $y_0 \geqslant a_n(1-\delta_n)$ then (2.12) is satisfied with $B = 1$. Suppose then, that

$$0 \leqslant y_0 < a_n(1-\delta_n).$$

Then (2.14) becomes

$$\left(1 - \frac{y_0}{a_n}\right) \leqslant C_4 \, \delta_n$$

which again implies (2.12) with $B = C_4$. ∎

Now, fix B in (2.12).

LEMMA 2.4. *Let $W \in \mathscr{E}$.*

(a) *Uniformly for $n \geqslant 1$, $1 \leqslant j \leqslant n$, and $x \in \mathbf{R}$,*

$$|l_{j,n}(U_n)(x)| \sim \frac{a_n^{3/2}}{n} \Psi_n W(x_{j,n}) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/4} \left|\frac{p_n(x)}{x - x_{j,n}}\right|. \tag{2.15}$$

(b) *There exists $C > 0$ such that uniformly for $n \geqslant 1$, $1 \leqslant j \leqslant n$, and $x \in \mathbf{R}$,*

$$|l_{j,n}(U_n)(x) \, W(x)| \, W^{-1}(x_{j,n}) \leqslant C. \tag{2.16}$$

(c) *Uniformly for $n \geqslant 1$ and $1 \leqslant j \leqslant n$,*

$$\frac{a_n^{3/2}}{n} \Psi_n(x_{j,n}) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/2} |p_n' W| \, (x_{j,n}) \sim a_n^{1/2} \, |p_{n-1} W| \, (x_{j,n}) \sim \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/4}. \tag{2.17}$$

(d) *For* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and* $|x| \leqslant a_n$, *there exists* $C > 0$ *such that*

$$|p_n(x)|\, W(x) \leqslant C \frac{n}{a_n^{3/2}} \left[\Psi_n(x)\, \Psi_n(x_{j,n}) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/2}\right]^{-1/2} \times |x - x_{j,n}|. \tag{2.18}$$

Proof. Parts (a), (b), and (c) are (2.13), (2.14),and (2.11) resp. in [5]. Part (d) is (10.28) in [8]. ∎

Lemma 2.5. *Let* $W \in \mathscr{E}$ *and let* $l_{n+1,n+2}(V_{n+2})$ *and* $l_{n+2,n+2}(V_{n+2})$ *be respectively the fundamental polynomials of degree* $\leqslant n+1$ *at the points* y_0 *and* $-y_0$. *Then there exists* $C > 0$ *such for all* $x \in \mathbf{R}$.

$$|l_{n+1,n+2}(V_{n+2})|\,(x)\, W(x)\, W^{-1}(y_0) \leqslant C \tag{2.19}$$

and

$$|l_{n+2,n+2}(V_{n+2})|\,(x)\, W(x)\, W^{-1}(-y_0) \leqslant C. \tag{2.20}$$

Proof. We prove (2.19). Relation (2.20) is similar. First observe that

$$l_{n+1,n+2}(V_{n+2})(x) = \frac{p_n(x)(y_0 + x)}{2y_0 p_n(y_0)} \in \mathscr{P}_{n+1} \tag{2.21}$$

and satisfies

$$l_{n+1,n+2}(V_{n+2})(y_0) = 1, \tag{2.22}$$

$$l_{n+1,n+2}(V_{n+2})(x_{j,n}) = 0, \qquad 1 \leqslant j \leqslant n \tag{2.23}$$

and

$$l_{n+1,n+2}(V_{n+2})(-y_0) = 0.$$

Observe that by (2.10), we may assume that $|x| \leqslant a_{n+1}$. Then by (2.6), (2.9), the definition of y_0, (2.12), and (2.21),

$$\begin{aligned} &|l_{n+1,n+2}(V_{n+2})\, W(x)\, W^{-1}(y_0)| \\ &\quad\leqslant C \frac{W(x)\, |p_n(x)|\, |y_0 + x|}{2y_0\, |p_n(y_0)|\, W(y_0)} \\ &\quad\leqslant C_1 \frac{a_n^{-1/2} n^{1/6} T(a_n)^{1/6}\, a_n}{2a_n(1 - B\delta_n)\, a_n^{-1/2} n^{1/6} T(a_n)^{1/6}} \leqslant C_2. \end{aligned}$$ ∎

We next need a lemma which gives an estimate of the distance between y_0 and $|x_{j,n}|$, $1 \leqslant j \leqslant n$.

LEMMA 2.6. *Let $W \in \mathscr{E}$. Then for $n \geqslant N_0$ and uniformly for $1 \leqslant j \leqslant n$, we have*

$$|y_0 - |x_{j,n}|| \sim a_n \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right). \tag{2.24}$$

Proof. We begin with our lower bound. We consider two cases:

Case 1. $|x_{j,n}| \geqslant a_n(1 - 2L\delta_n)$. Note that here

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \leqslant 3L\delta_n.$$

Moreover (2.1) implies

$$\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \leqslant 3L\delta_n \tag{2.25}$$

if L is large enough.

Next observe that by (2.12) and the definition of Ψ_n (see (1.16)), we have that

$$\psi_n^{1/2}(y_0) \geqslant (T(a_n)^{1/2}\, \delta_n^{1/4}(B + L)^{1/4})^{-1}. \tag{2.26}$$

Now as Q and $|p_n|$ are both even functions, the definition of Ψ_n, (1.16), (2.6), (2.18), (2.25), and (2.26) yield

$$\begin{aligned} |y_0 - |x_{j,n}|| &\geqslant C_1\, a_n\, \delta_n \\ &\geqslant C_2\, a_n \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right) \end{aligned}$$

uniformly for $1 \leqslant j \leqslant n$.

Case 2. $|x_{j,n}| \leqslant a_n(1 - 2L\delta_n)$. Observe that if L is large enough,

$$|y_0 - |x_{j,n}|| \geqslant a_n \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right) - (a_n(1 + L\delta_n) - y_0). \tag{2.27}$$

Now by (2.12),

$$(a_n(1 + L\delta_n) - y_0) \leqslant \frac{a_n}{2} \left[1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \right] \tag{2.28}$$

if

$$1-\frac{|x_{j,n}|}{a_n}\geqslant 2\delta_n\left[B+\frac{L}{2}\right]. \tag{2.29}$$

But then it is easy to see that $|x_{j,n}|\leqslant a_n(1-2L\delta_n)$ implies (2.29) if L is large enough and so we have (2.28). Inequality (2.27) then becomes

$$|y_0-|x_{j,n}||\geqslant\frac{a_n}{2}\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)$$

and we have our lower bound for this case as well.

The upper bound is easier. We again distinguish two cases:

Case 1. $|x_{j,n}|\leqslant a_n$. Here, if L is large enough, we have by (2.12),

$$\begin{aligned}|y_0-|x_{j,n}||&\leqslant L\,a_n\,\delta_n+a_n\left(1-\frac{|x_{j,n}|}{a_n}\right)\\&=a_n\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right).\end{aligned}$$

Case 2. $a_n\leqslant|x_{j,n}|\leqslant a_n(1+A\delta_n)$. Here if L is large enough, we have by (2.1) and (2.12),

$$\begin{aligned}|y_0-|x_{j,n}||&\leqslant B\,a_n\,\delta_n+x_{1,n}-a_n\\&\leqslant a_n\,\delta_n(B+A)\leqslant a_n\left[\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right].\end{aligned}$$

The lemma is proved. ∎

Let us put

$$\Delta x_{j,n}:=x_{j,n}-x_{j+1,n},\qquad 1\leqslant j\leqslant n.$$

We prove:

LEMMA 2.7. *Let* $W\in\mathscr{E}$, $n\geqslant N_0$, $r>1$, *and* $|x|\leqslant a_{rn}$. *Then there exists* $C_j>0$, $j=1,2$ *such that for* $1\leqslant j\leqslant n$,

(a) $W(x)\,l_{j,n}(U_n)(x)\,W^{-1}(x_{j,n})$

$$\leqslant C_1\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{1/4}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{-1/4}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}, \tag{2.30}$$

(b) $W(x)\, l_{j,n+2}(V_{n+2})(x)\, W^{-1}(x_{j,n})$

$$\leqslant C_1 \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right)^{-3/4} \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right)^{3/4} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}. \tag{2.31}$$

Proof. We begin first with (2.30). First note that (2.5) and (2.6) show that uniformly for n and x,

$$|p_n(x)|\, W(x) \leqslant C_1 a_n^{-1/2} \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right)^{-1/4}. \tag{2.32}$$

Then by (2.32),

$$\begin{aligned} &W(x)\, l_{j,n}(U_n)(x)\, W^{-1}(x_{j,n}) \\ &= \frac{W(x)\, |p_n(x)|\, W^{-1}(x_{j,n})}{|p_n'(x_{j,n})|\, |x - x_{j,n}|} \\ &\leqslant C_1 \frac{a_n^{-1/2}(|1 - (|x|/a_n)| + L\delta_n)^{-1/4}\, W^{-1}(x_{j,n})}{|p_n'(x_{j,n})|\, |x - x_{j,n}|} \\ &\leqslant C_2 \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right)^{1/4} \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right)^{-1/4} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}. \end{aligned}$$

by (2.2) and (2.17). So we have (2.30).

We now proceed with (2.31).

First observe that for $1 \leqslant j \leqslant n$,

$$l_{j,n+2}(V_{n+2})(x) = \left(\frac{y_0^2 - x^2}{y_0^2 - x_{j,n}^2} \right) l_{j,n}(U_n)(x). \tag{2.33}$$

Next, we claim that

$$|y_0 - x| \leqslant C_3\, a_n \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right). \tag{2.34}$$

We consider two cases:

Case 1. $|x| \leqslant a_n$. Here much as in the proof of Lemma 2.6,

$$\begin{aligned} |y_0 - |x|| &\leqslant B a_n \delta_n + a_n \left(1 - \frac{|x|}{a_n} \right) \\ &\leqslant C_3\, a_n \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right) \end{aligned}$$

if L is large enough.

Case 2. $a_n < |x| \leqslant a_{rn}$. Here, using (2.9),

$$\begin{aligned} |x| - a_n &\leqslant a_{rn} - a_n \\ &\leqslant C_4\, a_n T(a_n)^{-1} \\ &\leqslant C_5 a_n \left(\left|1 - \frac{|x|}{a_n}\right|\right) \end{aligned}$$

so that

$$\begin{aligned} |y_0 - |x|| &\leqslant |a_n - y_0| + |a_n - |x|| \\ &\leqslant C_6\, a_n \left(\left|1 - \frac{|x|}{a_n}\right| + L\delta_n\right) \end{aligned}$$

so (2.34) is established. Then (2.24), (2.30), (2.33), and (2.34) yield (2.31). ∎

3. THE PROOFS OF OUR UPPER BOUNDS

In this section we establish our upper bounds for (1.17) and (1.24). Throughout we assume that $W \in \mathscr{E}$, $x \in \mathbf{R}$ is fixed, and $x_{k(x),\, n}$ is that zero of p_n closest to x.

We need two lemmas

LEMMA 3.1. *There exist M and $\delta > 0$ with the following properties:*

(a) *If* $|x| \in [0, a_n(1 + (L/2)\delta_n)]$ *then*

(i) $$\{j\colon |j - k(x)| \leqslant 2\} \subseteq \left\{ j\colon |x - x_{j,\, n}| \leqslant \frac{Ma_n}{n}\, \Psi_n(x) \right\}. \tag{3.1}$$

(ii) $$|x - x_{k(x) \pm k,\, n}| \leqslant \delta\, \frac{a_n}{n}\, \Psi_n(x), \qquad k = 0, 1.$$

(iii) $$|x - x_{k(x) \pm 3,\, n}| > \frac{Ma_n}{n}\, \Psi_n(x). \tag{3.2}$$

(b) *If* $|x| \in [a_n(1 + (L/2)\delta_n), \infty)$,

$$|x - x_{j,\, n}| > \frac{Ma_n}{n}\, \Psi_n(x) \tag{3.3}$$

for all $1 \leqslant j \leqslant n$.

Proof. Suppose first that $x \in [0, a_n(1+(L/2)\delta_n)]$. Observe that if $t \in [x_{j+1,n}, x_{j,n}]$, $1 \leqslant j \leqslant n$, we have

$$\left| \frac{1-(|t|/a_n)+L\delta_n}{1-(|x_{j,n}|/a_n)+L\delta_n} - 1 \right| \leqslant \frac{1}{a_n} \left| \frac{x_{j,n}-t}{1-(|x_{j,n}|/a_n)+L\delta_n} \right|$$
$$\leqslant \frac{1}{a_n} \left| \frac{x_{j,n}-x_{j+1,n}}{1-(|x_{j,n}|/a_n)+L\delta_n} \right| \leqslant \frac{C\Psi_n(x_{j,n})}{n(L-A)\,\delta_n} \leqslant \frac{1}{2} \tag{3.4}$$

by (1.16), (2.1), and (2.2) if L is large enough.

We conclude using (1.16) and (3.4) that

$$\Psi_n(t) \sim \Psi_n(x_{j,n}) \qquad \text{uniformly for} \quad j, n \qquad \text{and} \qquad t \in [x_{j+1,n}, x_{j,n}]. \tag{3.5}$$

Now by definition of $x_{k(x),n}$, we must have $x \in [x_{k(x)+1,n}, x_{k(x),n}]$ or $x \in [x_{k(x),n}, x_{k(x)-1,n}]$ at least when $x \leqslant x_{1,n}$. Using (2.3) and (2.4) if necessary, we may assume without loss of generality that $x \in [x_{k(x)+1,n}, x_{k(x),n}]$.

Then by (2.2) and (3.5),

$$|x - x_{k(x)\pm 2,n}| \leqslant |x_{k(x)-2,n} - x_{k(x)+2,n}|$$
$$\leqslant C \frac{a_n}{n} \Psi_n(x_{k(x),n})$$
$$\sim \frac{a_n}{n} \Psi_n(x). \tag{3.6}$$

Using (3.6) and (2.2) we see that it is possible to choose M such that (3.1) holds at least when $x \leqslant x_{1,n}$. Suppose $x \geqslant x_{1,n}$. We may then suppose that L is chosen large enough such that $x_{3,n} \geqslant a_n(1-(L/4)\delta_n)$ and then

$$|x - x_{3,n}| \leqslant a_n\left(1+\frac{L}{2}\delta_n\right) - a_n\left(1-\frac{L}{4}\delta_n\right)$$
$$\sim a_n\,\delta_n \sim \frac{a_n}{n} \Psi_n(x)$$

using (1.15) and (1.16).

Thus also in this case, it is possible to choose M such that (3.1) holds. Parts (ii) and (iii) of the lemma then follow similarly. ∎

Now fix M and δ in Lemma 3.1 and put

$$J_n := [x_{n,n}, x_{1,n}] \backslash [x_{k(x)+2}, x_{k(x)-2}] \tag{3.7}$$

if $|x| \in [0, a_n(1+(L/2)\delta_n)]$ and

$$J_n := [x_{n,n}, x_{1,n}] \tag{3.8}$$

if $|x| \in [a_n(1+(L/2)\delta_n, \infty)$.

We modify the definition in (3.7) accordingly if $|x| \geqslant x_{1,n}$.

We have the following estimate.

LEMMA 3.2. *Uniformly for* $1 \leqslant j \leqslant n$ *and* $n \geqslant N_0$,

$$\sum_{\substack{j=1 \\ j \notin [k(x)+2,\, k(x)-2]}}^{n} \frac{\Delta x_{j,n}}{|x-x_{j,n}|^\alpha} = \left\{ \begin{array}{ll} O(a_n^{1-\alpha}), & 0<\alpha<1 \\ O(\log n), & \alpha=1 \\ O\left(\dfrac{n}{a_n \Psi_n(x)}\right)^{\alpha-1}, & \alpha>1. \end{array} \right\} \tag{3.9}$$

Proof. First note that if $|x| \leqslant a_n(1+(L/2)\delta_n)$, we have uniformly for $n \geqslant N_0$ and $1 \leqslant j \leqslant n$,

$$|x-t| \sim |x-x_{j,n}|, \qquad t \in [x_{j+1,n}, x_{j,n}],\ j \notin [k(x)+2, k(x)-2]. \tag{3.10}$$

This follows much as in [3] using Lemma 3.1(a) and (2.2) since,

$$\left|\frac{x-t}{x-x_{j,n}} - 1\right| = \left|\frac{t-x_{j,n}}{x-x_{j,n}}\right| \leqslant \left|\frac{x_{j,n}-x_{j+1,n}}{x-x_{j,n}}\right| \leqslant C$$

and similarly we can bound

$$\frac{x-x_{j,n}}{x-t}.$$

Then, from (2.2) and the definition of J_n in (3.7), we obtain

$$\sum_{\substack{j=1 \\ j \notin [k(x)+2,\, k(x)-2]}}^{n} \frac{\Delta x_{j,n}}{|x-x_{j,n}|^\alpha} = O\left(\int_{\substack{|t| \leqslant a_n(1+A\delta_n) \\ t \in J_n}} \frac{dt}{|x-t|^\alpha}\right) = \left\{ \begin{array}{ll} O(a_n^{1-\alpha}), & 0<\alpha<1 \\ O(\log n), & \alpha=1 \\ O\left(\dfrac{n}{a_n \Psi_n(x)}\right)^{\alpha-1}, & \alpha>1 \end{array} \right\}.$$

The case for $|x| \geqslant a_n(1+(L/2)\delta_n)$ is similar but easier. ∎

We may now proceed with the proofs of our upper bounds. We begin with:

Proof of the Upper Bound in (1.17). From (2.30) we have for $1 \leqslant j \leqslant n$,

$$W(x)\, l_{j,n}(U_n)(x)\, W^{-1}(x_{j,n}) \leqslant C_1 \left(\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n} \right)^{1/4} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}.$$

Thus, by (1.6) and using the above, we have

$$\begin{aligned}\Lambda_n(W, U_n)(x) &= \sum_{j=1}^{n} W(x)\, |l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n}) \\ &\leqslant \sum_{j \in [k(x)+2,\, k(x)-2]} W(x)\, |l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n}) \\ &\quad + C_1 \sum_{j \notin [k(x)+2,\, k(x)-2]} \left(\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n} \right)^{1/4} \\ &\quad \times \frac{\Delta x_{j,n}}{|x-x_{j,n}|}. \end{aligned} \tag{3.11}$$

First observe that we may write

$$\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n} \leqslant 1 + \frac{|x-x_{j,n}|}{a_n(|1-(|x|/a_n)| + L\delta_n)}. \tag{3.12}$$

Next we observe that using (2.10), we may assume without loss of generality that $|x| \leqslant a_n$. Then (3.12) becomes using the definition of δ_n (see (1.15))

$$\left(\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n} \right)^{1/4} = O(1) + O\left(\frac{n^{1/6} T(a_n)^{1/6}\, |x-x_{j,n}|^{1/4}}{a_n^{1/4}} \right). \tag{3.13}$$

Thus using (2.16), (3.9), and (3.13), we now rewrite (3.11) as

$$\begin{aligned}\Lambda_n(W, U_n)(x) &\leqslant C_2 \sum_{j \in [k(x)+2,\, k(x)-2]} 1 \\ &\quad + O\left(\sum_{j \notin [k(x)+2,\, k(x)-2]} \frac{n^{1/6} T(a_n)^{1/6}\, \Delta x_{j,n}}{a_n^{1/4}\, |x-x_{j,n}|^{3/4}} \right) \\ &\quad + O\left(\sum_{j \notin [k(x)+2,\, k(x)-2]} \frac{\Delta x_{j,n}}{|x-x_{j,n}|} \right) \\ &= O(1) + O(\log n) + O(n^{1/6} T(a_n)^{1/6}) \\ &= O(n^{1/6} T(a_n)^{1/6}) \end{aligned} \tag{3.14}$$

and so we have taking sup's,

$$\|\Lambda_n(W, U_n)\|_{L_\infty(\mathbf{R})} = O(n^{1/6}T(a_n)^{1/6}) \tag{3.15}$$

as required. ∎

We now present,

Proof of Our Upper Bound in (1.24). Firstly, from (2.31) we have for $1 \leqslant j \leqslant n$,

$$W(x)\, l_{j,n+2}(V_{n+2})(x)\, W^{-1}(x_{j,n}) \leqslant C_1 \left(\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n}\right)^{-3/4} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}. \tag{3.16}$$

Thus by (1.6), (2.19), (2.20), and (3.16), we have

$$\begin{aligned} &\Lambda_{n+2}(W, V_{n+2})(x) \\ &\leqslant O(1) + \sum_{\substack{j=1 \\ j\in[k(x)+2,\, k(x)-2]}}^{n} W(x)\, |l_{j,n+2}(V_{n+2})(x)|\, W^{-1}(x_{j,n}) \\ &\quad + C_2 \sum_{\substack{j=1 \\ j\notin[k(x)+2,\, k(x)-2]}}^{n} \left(\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n}\right)^{-3/4} \frac{\Delta x_{j,n}}{|x-x_{j,n}|} \\ &= O(1) + \sum_1 (x) + \sum_2 (x), \end{aligned} \tag{3.17}$$

where

$$\sum_1 (x) := \sum_{\substack{j=1 \\ j\in[k(x)+2,\, k(x)-2]}}^{n} W(x)\, |l_{j,n+2}(V_{n+2})(x)|\, W^{-1}(x_{j,n})$$

and

$$\sum_2 (x) := C_2 \sum_{\substack{j=1 \\ j\notin[k(x)+2,\, k(x)-2]}}^{n} \left(\frac{|1-(|x_{j,n}|/a_n)| + L\delta_n}{|1-(|x|/a_n)| + L\delta_n}\right)^{-3/4} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}.$$

We observe that using (2.11), we may assume without loss of generality that $|x| \leqslant a_{n+1}$. We begin with the estimation of $\sum_1 (x)$.

Note, that by (2.24), (2.33), and (2.34),

$$\sum_1 (x) = \sum_{\substack{j=1 \\ j \notin [k(x)+2,\, k(x)-2]}}^{n} \left| \frac{y_0^2 - x^2}{y_0^2 - x_{j,n}^2} \right| W(x)\, |l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n})$$

$$= O\left(\sum_{\substack{j=1 \\ j \in [k(x)+2,\, k(x)-2]}}^{n} \left(\frac{|1-(|x|/a_n)| + L\delta_n}{|1-(|x_{j,n}|/a_n)| + L\delta_n} \right) \right.$$

$$\left. \times W(x)\, |l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n}) \right). \tag{3.18}$$

Next, using (2.1), (2.2), and (2.4), it is easy to see that if L is large enough, we have uniformly for x and $j \in [k(x)+2, k(x)-2]$,

$$\left(\frac{|1-(|x|/a_n)| + L\delta_n}{|1-(|x_{j,n}|/a_n)| + L\delta_n} \right) \sim 1$$

so that

$$\sum_1 (x) = O\left(\sum_{\substack{j=1 \\ j \in [k(x)+2,\, k(x)-2]}}^{n} W(x)\, |l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n}) \right)$$

$$= O(1) \tag{3.19}$$

by (2.16).

We now turn to the delicate estimation of $\sum_2 (x)$.

Much as in (3.12), we observe that for $1 \leqslant j \leqslant n$ we have

$$\left(\frac{|1-(|x|/a_n)| + L\delta_n}{|1-(|x_{j,n}|/a_n)| + L\delta_n} \right)^{3/4}$$

$$\leqslant 1 + \frac{|x - x_{j,n}|^{3/4}}{a_n^{3/4} (|1-(|x_{j,n}|/a_n)| + L\delta_n)^{3/4}}. \tag{3.20}$$

Then, using (3.20), we may write

$$\sum_2 (x) = O\left(\sum_{j \in S} \frac{\Delta x_{j,n}}{|x - x_{j,n}|} + \sum_{j \in S} \frac{\Delta x_{j,n}}{a_n^{3/4}\, |x - x_{j,n}|^{1/4}\, (|1-(|x_{j,n}|/a_n)| + L\delta_n)^{3/4}} \right),$$

where

$$S = \{ j \colon 1 \leqslant j \leqslant n,\ j \notin [k(x)+2, k(x)-2] \},$$

$$= O(\log n) + O\left(\sum_{j \in S} \frac{\Delta x_{j,n}}{|x - x_{j,n}|^{1/4}\, (|a_n - |x_{j,n}|| + a_n L\delta_n)^{3/4}} \right)$$

by (3.9)

$$= O(\log n) + O\left(\sum_{\substack{j \in S \\ |x_{j,n}| \leqslant a_n(1-\delta_n)}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|^{1/4} (a_n - |x_{j,n}|)^{3/4}} \right)$$
$$+ O\left(\sum_{\substack{j \in S \\ |x_{j,n}| > a_n(1-\delta_n)}} \frac{\Delta x_{j,n} n^{1/2} T(a_n)^{1/2}}{|x - x_{j,n}|^{1/4} a_n^{3/4}} \right). \quad (3.21)$$

Next, using the Geometric and Arithmetic mean inequality and (3.9) again, we may continue (3.21) as

$$\sum_2 (x) = O(\log n)$$
$$+ O\left(\sum_{\substack{j \in S \\ |x_{j,n}| \leqslant a_n(1-\delta_n)}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|} \right) + O\left(\sum_{\substack{j \in S \\ |x_{j,n}| \leqslant a_n(1-\delta_n)}} \frac{\Delta x_{j,n}}{a_n - |x_{j,n}|} \right)$$
$$+ O\left(\frac{n^{1/2} T(a_n)^{1/2}}{a_n^{3/4}} \sum_{\substack{j \in S \\ |x_{j,n}| > a_n(1-\delta_n)}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|^{1/4}} \right)$$
$$= O(\log n) + O\left(\sum_{\substack{j \in S \\ |x_{j,n}| > a_n(1-\delta_n)}} 1 \right), \quad (3.22)$$

where in the last line we used (1.15), (1.16), (2.1), and (2.2).

Now it remains to observe that the spacing (2.2) and (1.16) imply that there exist at most a finite number of j such that $|x_{j,n}| > a_n(1-\delta_n)$. Then (3.22) yields

$$\sum_2 (x) = O(\log n) + O(1) = O(\log n). \quad (3.23)$$

Combining (3.23) with (3.19) and taking sup's yield

$$\|\Lambda_{n+2}(W, V_{n+2})\|_{L_\infty(\mathbf{R})} = O(\log n) \quad (3.24)$$

as required. ∎

4. THE PROOFS OF THEOREMS 1.2 AND 1.4 AND COROLLARIES 1.3 AND 1.5

In this section we present the proofs of our lower bounds in (1.17) and (1.24). We deduce Theorems 1.2 and 1.4 and Corollaries 1.3 and 1.5.

We begin with,

Proof of Our Lower Bound in (1.17). Write

$$\Lambda_n(W, U_n)(x) = W(x)\,|p_n(x)| \sum_{j=1}^{n} p_n'(x_{j,n})^{-1}\, W(x_{j,n})^{-1}\, |x - x_{j,n}|^{-1}. \quad (4.1)$$

In particular, (4.1) becomes using (1.16), (2.9), (2.12), and (2.17),

$$\begin{aligned}\Lambda_n(W, U_n)(y_0) &\geqslant C_1 \sum_{0 \leqslant x_{j,n} \leqslant a_n/2} \frac{a_n^{-1/2} n^{1/6} T(a_n)^{1/6}}{a_n^{-1/2}(1 - (|x_{j,n}|/a_n) + L\delta_n)^{-1/4}\, n} \\ &\geqslant C_2 n^{-5/6} T(a_n)^{1/6} \sum_{0 \leqslant x_{j,n} \leqslant a_n/2} 1. \quad (4.2)\end{aligned}$$

Now it remains to observe that the spacing (2.2) and (1.16) imply that there exist $\geqslant C_3 n\ j$ such that $x_{j,n} \in [0, a_n/2]$. Then (4.2) becomes

$$\Lambda_n(W, U_n)(y_0) \geqslant C_4 n^{1/6} T(a_n)^{1/6}$$

so that

$$\|\Lambda_n(W, U_n)\|_{L_\infty(\mathbf{R})} \geqslant \Lambda_n(W, U_n)(y_0) \geqslant C_5 n^{1/6} T(a_n)^{1/6}, \quad (4.3)$$

as required. ∎

We now turn to the proof of our lower bound (1.24). Here a choice of $x = y_0$ is not sufficient to achieve our lower bound and we need to proceed more carefully. Indeed, we will show that the point we need sits "far" away from a_n.

Proof of Our Lower Bound for (1.24). First we claim that there exists $y \in \mathbf{R}$ satisfying $|y| \leqslant \alpha a_n$, for some $0 < \alpha < 1$ and uniformly for $n \geqslant 1$,

$$a_n^{1/2} p_n W(y) \sim 1. \quad (4.4)$$

To see this, observe first that if $0 < \alpha < 1$ is given, then by (1.16), (2.2), and (2.9), there exits $C_1 n > j$, $1 \leqslant j \leqslant n+1$ such that $|x_{j,n+1}| \in [0, \alpha a_n]$. Now choose $y = y_1 = x_{k,n+1}$ for some $1 \leqslant k \leqslant n+1$ such that $|y_1| \in [0, \alpha a_n]$. Then (2.9) and (2.17) give

$$a_n^{1/2}\, |p_n W|\, (y_1) \sim 1$$

and (4.4) is established. Fix y_1 as above.

We now proceed as follows. Since $y_1 < c y_0$, for some $0 < c < 1$, we have by (1.29), (2.12), (2.33), and (4.4),

$$\begin{aligned}\Lambda_{n+2}(W, V_{n+2})(y_1) &\geqslant \sum_{j=1}^{n} W(y_1)\, W(x_{j,n})^{-1} \left(\frac{y_0^2 - y_1^2}{y_0^2}\right) l_{j,n}(U_n)(y_1)\\ &\geqslant C_1 \sum_{j=1}^{n} W(y_1)\, W(x_{j,n})^{-1}\, l_{j,n}(U_n)(y_1)\\ &\geqslant C_2 \Lambda_n(W, U_n)(y_1) \geqslant C_3\, a_n^{1/2}\, |p_n W|\,(y_1) \log n\\ &\geqslant C_4 \log n.\end{aligned}$$

Thus,

$$\|\Lambda_{n+2}(W, V_{n+2})\|_{L_\infty(\mathbf{R})} \geqslant C_4 \log n \tag{4.5}$$

and we have proved our lower bound. ∎

We may now present:

Proof of Theorem 1.2. This follows immediately from (3.15) and (4.3). ∎

Proof of Corollary 1.3. Relation (1.18) follows from the representation (1.6), (1.17), and Theorem 1.2 of [4]. Relations (1.21) and (1.22) follow from (1.18), Corollary 1.7 of [3], and (2.7). ∎

Proof of Theorem 1.4. This follows immediately from (3.24) and (4.5). ∎

Proof of Corollary 1.5. Relation (1.25) follows from the representation (1.6), (1.24), and Theorem 1.2 of [4]. Relations (1.26) and (1.27) follow from (1.25), Corollary 1.7 of [3], and (2.7). ∎

5. POINTWISE ESTIMATES OF $\Lambda_n(W, U_n)$

In this section, we sketch briefly the proof of Theorem 1.6. Fix x, $x_{k(x),n}$, M, δ, and J_n as in Section 3.

Step 1. Set

$$S_1 := \left\{ j\colon \leqslant j \leqslant n,\ |x - x_{j,n}| \leqslant \frac{\delta a_n}{n}\, \Psi_n(x) \right\},$$

$$S_2 := \left\{ j\colon \leqslant j \leqslant n,\ \frac{\delta a_n}{n}\, \Psi_n(x) \leqslant |x - x_{j,n}| \leqslant \frac{M a_n}{n}\, \Psi_n(x) \right\},$$

and

$$S_3 := \left\{ j \colon 1 \leqslant j \leqslant n,\ |x - x_{j,n}| > \frac{Ma_n}{n}\, \Psi_n(x) \right\}.$$

Now write

$$\Lambda_n(U_n, W)(x) := \sum_{j \in S_1} (x) + \sum_{j \in S_2} (x) + \sum_{j \in S_3} (x).$$

Step 2. *Estimation of* $\sum_{j \in S_1} (x)$ *and* $\sum_{j \in S_2} (x)$. First observe that it suffices to estimate the above sums for $x \in [0, a_n(1 + (L/2)\delta_n)]$ for they are identically zero outside this range of x. Moreover, recall that we may assume by symmetry that $x > 0$.

Then the following holds:

LEMMA 5.1. *Let* $W \in \mathscr{E}$.

(a) *There exists* $C_1 \geqslant 0$ *such that uniformly for* $n \geqslant 1$ *and* $x \in [0, a_n(1 + (L/2)\delta_n)]$,

$$0 \leqslant \sum_{j \in S_1} (x) \leqslant C_1. \tag{5.1}$$

Moreover, uniformly for $n \geqslant 1$ *and* $x \in [0, x_{1,n}]$,

$$\sum_{j \in S_1} (x) \sim 1. \tag{5.2}$$

(b) *Uniformly for* $x \in [0, a_n(1 + (L/2)\delta_n)]$ *and* $n \geqslant N_0$,

$$\sum_{j \in S_2} (x) \sim \sqrt{a_n}\, |p_n W|\, (x) \left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{1/4}. \tag{5.3}$$

Proof. First note that (2.16) gives

$$\begin{aligned}\sum_{j \in S_1} (x) &= W(x) \sum_{j \in S_1} |l_{j,n}(U_n)(x)|\ W^{-1}(x_{j,n}) \\ &\leqslant C \sum_{j \in S_1} 1 \leqslant C_1\end{aligned}$$

for some $C_1 > 0$ independent of x and n as the above sum is finite.

For the lower sum, we use the weighted Erdős–Turan inequality (see, for example, [5])

$$l_{j,n}(U_n)(x)\, W(x)\, W^{-1}(x_{j,n}) + l_{j+1,n}(U_n)(x)\, W(x)\, W^{-1}(x_{j+1,n}) \geqslant 1 \tag{5.4}$$

valid for $n \geqslant 2$, $1 \leqslant j \leqslant n-1$, and $x \in [x_{j+1,n}, x_{j,n}]$.

If $x \leqslant x_{1,n}$, we may assume without loss of generality that $x \in [x_{k(x)+1,n}, x_{k(x),n}]$. Then (5.4) gives

$$\sum_{j \in S_1} (x) \geqslant W(x) \sum_{j=k(x)}^{k(x)+1} l_{j,n}(U_n)(x)\, W^{-1}(x_{j,n}) \geqslant C_2.$$

Thus (5.1) and (5.2) follow.

It remains to show (5.3). Here we first observe that by (2.2) we have uniformly for $j \in S_2$,

$$\frac{a_n}{n} \Psi_n(x_{j,n}) \sim |x_{j,n} - x_{j\pm 1,n}| \sim |x - x_{j,n}|. \tag{5.5}$$

Then (2.15) and (5.5) easily yield.

$$\sum_{j \in S_2} (x) \sim \sqrt{a_n}\, |p_n W|\, (x) \left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{1/4}$$

as required. ∎

Step 3. *Preliminary Estimation of* $\sum_{j \in S_3} (x)$.

LEMMA 5.2. *Let* $W \in \mathscr{E}$.

(a) *If* $|x| \leqslant 2a_n$, *we have uniformly for* x *and* $n \geqslant N_0$,

$$\sum_{j \in S_3} (x) \sim \sqrt{a_n}\, p_n W(x) \int_{\substack{|t| \leqslant a_n(1+A\delta_n) \\ t \in J_n}} \frac{(1 - (|t|/a_n) + L\delta_n)^{1/4}}{|x-t|}\, dt. \tag{5.6}$$

(b) *If* $|x| \leqslant 2a_n$, *we have uniformly for* x *and* $n \geqslant N_0$,

$$\sum_{j \in S_3} (x) \sim \frac{\sqrt{a_n}\, p_n W(x)}{|x|} \int_{\substack{|t| \leqslant a_n(1+A\delta_n) \\ t \in J_n}} \left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{1/4} dt. \tag{5.7}$$

Proof. We consider the case $x \in [0, a_n(1 + (L/2)\delta_n)]$ and $x \leqslant x_{1,n}$. The other cases are similar.

By (2.2) and (2.15),

$$\sum_{j\in S_3}(x)\sim\sqrt{a_n}\,p_n W(x)\times\sum_{j\in[1,n]\backslash[k(x)+2,\,k(x)-2]}\int_{x_{j+1,n}}^{x_{j,n}}\frac{(1-(|x_{j,n}|/a_n+L\delta_n)^{1/4}}{|x-x_{j,n}|}\,dt. \tag{5.8}$$

Then much as in (3.10), (5.8) readily yields (5.6) for this case. ▮

Step 4. *Estimation of* $J:=\int_{|t|\leqslant a_n(1+A\delta_n),\,t\in J_n}(1-|t|/a_n+L\delta_n)^{1/4}\,dt$. We now record the following technical estimate for J:

LEMMA 5.3. *Let* $W\in\mathscr{E}$ *and suppose that* $x\in[0,a_n(1+(L/2)\delta_n)]$. *Then uniformly for* x *and* $n\geqslant N_0$,

$$J\sim\left(1-\frac{|x|}{a_n}+L\delta_n\right)^{1/4}\log\left(\frac{n(1-|x|/a_n+L\delta_n)}{\Psi_n(x)}\right)+1. \tag{5.9}$$

Step 5. *Proof of Theorem* 1.6. Observe that for $|x|\leqslant a_n(1+(L/2)\delta_n)$,

$$\log\left(\frac{n(1-|x|/a_n+L\delta_n)}{\Psi_n(x)}\right)>0 \qquad \text{if}\quad L \text{ is large enough.}$$

Then (5.1), (5.2), (5.3), and (5.9) yield the result for this case. Theorem 1.6(b) and (c) are similar but easier. ▮

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